Closing Wed: HW_8A,8B,8C(8.1,9.1)
Closing *Next* Wed: HW_9A, 9B (9.3, 9.4)

9.1 Intro to Differential Equations

A **differential equation** is an equation involving derivatives. A **solution to a differential equation** is any function that satisfies the equation.

Entry Task: Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer

Example

Consider the differential equation:

$$\frac{dP}{dt} = 2P$$

(a) Is $P(t) = 8e^{2t}$ a solution?

(b) Is $P(t) = t^3$ a solution?

(c) Is $P(t) = 0$ a solution?

Question: What can you say about the behavior of the solutions without solving?

The **general solution** to

$$\frac{dP}{dt} = 2P$$

is

$$P(t) = Ce^{2t},$$

for any constant C.

We will learn how to find this next time.

Initial conditions note:

If you are asked to find the solutions to

$$\frac{dP}{dt} = 2P \quad \text{with} \quad P(0) = 12$$

then

Step 1: Find the general solution

$$P(t) = Ce^{2t}$$

Step 2: Plug in initial condition

$$P(0) = Ce^{2(0)} = 12$$

$$C\,(1) = 12$$

$$C = 12$$

Thus,

$$P(t) = 12e^{2t}$$

Check!!!

Example: Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

(b) Is $y = t\,e^{-t}$ a solution?

(c) There is a sol'n that looks like $y = e^{rt}$. Can you find the value of *r* that works?

Application Tools:

$\frac{dy}{dt}$ = “instantaneous **rate of change** of *y* with respect to *t* ”

“A is proportional to B” means

A = kB, where k is a constant.

In other words, A/B = k.

1. Natural Unrestricted population

Assumption: "*The rate of growth of a population is proportional to the size of the population*."

P(t) = the population at year *t*,

$\frac{dP}{dt} =$ the rate of change of the population with respect to time (i.e. rate of growth).

The assumption is equivalent to

$$\frac{dP}{dt} = kP,$$

for some constant *k.*

2. Newton's Law of Cooling

Assumption: "*The rate of cooling is proportional to the temperature difference between the object and its surroundings*."

T_s = constant temp. of surroundings

$T(t) =$ temp. of the object at time t,

$\frac{dT}{dt}$ = rate of change of temp. with respect to time.

$T - T_s$ = temp. difference between object and surroundings.

Newton's Law of Cooling states

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant k.

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water.

A salt water mixture (brine) is being dumped **into** the vat at 2 gal/min and this mixture contains 3 grams of salt per gal. The vat is thoroughly mixed. At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let y(t) = grams of salt in vat at time *t*.

$\frac{dy}{dt} =$ the rate (g/min) at which salt is changing with respect to time.

Note:

$\frac{y(t)}{50} =$ *salt per gallon* at time, *t*.

RATE IN = $\left(3\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = 6\,\frac{\text{g}}{\text{min}}$

RATE OUT = $\left(\frac{y}{50}\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = \frac{y}{25}\,\frac{\text{g}}{\text{min}}$

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25} \quad \text{with } y(0) = 0.$$

4. All motion problems!

Consider an object of mass *m* kg moving up and down.

Let y(t) = `height at time *t*'

$\frac{dy}{dt}$ = `velocity at time *t*'

$\frac{d^2y}{dt^2}$ = `acceleration at time *t*'

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m\frac{d^2y}{dt^2} = \text{sum of forces on object}$$

Assumption 1:

Only force is force due to gravity (no air resistance):

$$m\frac{d^2y}{dt^2} = -mg$$

Assumption 2:

Consider gravity and ***air resistance***. Assume the force due to air resistance is proportional to velocity and in the opposite direction of velocity. Then

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$

5. Many, many others:

Random Example:

A common assumption for melting snow/ice is "the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area."

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for *r*.